ON THE SOLUTION OF PROBLEMS OF NON-STATIONARY CREEP

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Great difficulties arise in connection with the solution of problems of non-stationary creep. A method of numerical solution was considered recently by Kuratov and Rozenblium [1]. The author had earlier proposed a simple approximate method [2,3]. That method was based on the satisfaction of the minimum of the rate of complementary energy of the body

$$\Lambda + \frac{\partial \Pi}{\partial t} = \min \tag{1}$$

in the form (for the fundamental problem)

$$\sigma_{ij} = \sigma_{ij}' + \tau(t)(\sigma_{ij}'' - \sigma_{ij}') \equiv \sigma_{ij}^{(0)} \qquad (i,\ j = 1,\ 2,\ 3) \tag{2}$$

Here and in the following Λ is the complementary rate of dissipation, Π is the elastic strain energy, t is time, σ_{ij} are components of stress, σ_{ij}' and σ_{ij}'' are the components of stress of the problems of elasticity and of stationary creep, respectively. The function $\tau(t)$ increases monotonically and approaches 1 for $t \to \infty$.

Analogous solutions are constructed for the relaxation and mixed problems [2,3].

The solution (2) is, in fact, the first approximation. In this note a method of construction of closer approximations is outlined.

We assume the solution of problem (1) in the form

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sum_{k=1,2,\ldots} c_k(t)\, \sigma_{ij}^{(k)} \tag{3}$$

where $\sigma_{ij}^{(k)}$ are particular solutions of the homogeneous equations of equilibrium satisfying homogeneous boundary conditions on S_F, $c_k = c_k(t)$

are arbitrary functions of time. The solution $\sigma_{ij}^{(0)}$ is singled out, since it proves to be a good approximation. The rate of complementary energy of the body now becomes a function of τ and c_k, and the condition of minimum leads to the following system of differential equations with respect to c_k:

$$B(t)\frac{\partial \Lambda_1}{\partial c_i}+\frac{\partial^2 \Pi}{\partial \tau\,\partial c_i}\frac{d\tau}{dt}+\sum_{k=1,2,\ldots}\frac{\partial^2 \Pi}{\partial c_i \partial c_k}\frac{dc_k}{dt}=0 \qquad (i=1,2,\ldots) \tag{4}$$

For the sake of simplicity, it is assumed that the creep curves are affine and thus $\Lambda = B(t)\,\Lambda_1$, where Λ_1 is a function of stress only. The initial conditions for c_k are obvious:

$$c_k = 0 \quad \text{for} \quad t = 0 \tag{5}$$

Since Π is a positive quadratic form of τ and c_k, the second derivatives in Equations (4) are constant, and the determinant is

$$\Delta \equiv \left|\frac{\partial^2 \Pi}{\partial c_i\,\partial c_k}\right| > 0 \tag{6}$$

With condition (6), the system (4) can be reduced to the normal form

$$\frac{dc_i}{dt} = -\frac{1}{\Delta}\sum_{k=1,2,\ldots}\Delta_{ik}\left[B(t)\frac{\partial \Lambda_1}{\partial c_k}+\frac{\partial^2 \Pi}{\partial \tau \partial c_k}\frac{d\tau}{dt}\right] \quad (i=1,2,\ldots) \tag{7}$$

where Δ_{ik} is the cofactor of the corresponding element of the determinant Δ.

If the solution of the variational equation of stationary creep $\delta\Lambda_1 = 0$ (for the same body and the same loading) is sought in the form (3), then the values

$$\tau = 1, \quad c_i = 0 \qquad (i = 1, 2, \ldots) \tag{8}$$

correspond to the exact solution; it is then

$$\frac{\partial \Lambda_1}{\partial c_i} = 0; \qquad \delta^2\Lambda_1 = \sum_{i,\,j=1,2,\ldots}\frac{\partial^2 \Lambda_1}{\partial c_i \partial c_j}\delta c_i \delta c_j > 0 \tag{9}$$

Note that, repeating the considerations of Hill [4], it can be shown that for $\sigma_{ij} = \sigma_{ij}''$ the complementary dissipation Λ_1 becomes an absolute minimum, and this solution is unique.

We now indicate some properties of the functions $c_k(t)$ for larger times. In the fundamental problem considered, the body is subjected to the action of finite and constant-in-time loading; therefore, the complementary dissipation is bounded, and the stresses are bounded in a sufficiently large volume. Consequently, the functions $c_k(t)$ are also bounded.

Equation (7) can be written in integral form

$$-\int_0^{c_i} \Big(\sum_{k=1,2,\ldots} \Delta_{ik} \left[B(t) \frac{\partial \Lambda_1}{\partial c_k} + \frac{\partial^2 \Pi}{\partial \tau \partial c_k} \frac{d\tau}{dt} \right] \Big)^{-1} dc_i = \frac{1}{\Delta} \int_0^t dt \qquad (i = 1, 2, \ldots) \tag{10}$$

For $t \to \infty$ the right-hand sides of the relations (10) increase without bounds, and

$$\tau \to 1, \quad \frac{d\tau}{dt} \to 0, \quad B(t) \to B > 0$$

Since c_i are bounded it is necessary that for $t \to \infty$

$$\sum_{k=1,2,\ldots} \Delta_{ik} \frac{\partial \Lambda_1}{\partial c_k} = 0 \qquad (i = 1, 2, \ldots)$$

The determinant $|\Delta_{ik}|$ of this system is reciprocal to the determinant Δ and, consequently, different from zero. Therefore, $\partial \Lambda_1 / \partial c_k = 0$ and this implies $c_k = 0$. Because it also follows from (7) that $dc_k/dt \to 0$, the functions $c_k(t)$ approach zero asymptotically for $t \to \infty$.

The solution of the system (9) can be obtained numerically (or graphically) by the method of Euler.

The method presented here can easily be generalized to relaxation and mixed problems, the problems of a non-uniformly heated body, non-affine curves of creep, and elastic-plastic deformations.

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